JOURNAL OF COMBINATORIAL THEORY, Series A 24, 141–145 (1978)

The Four Biplanes with $k = 9$

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Communicated by Peter J. Cameron

Received November 3, 1976

An exhaustive computer search has established the existence of precisely four biplanes with $k = 9$. One of the self-dual biplanes is new and a description is provided in terms of its automorphism group. The λ-chain structures for all four biplanes are tabulated.

1. INTRODUCTION

Only finitely many biplanes (symmetric designs with $\lambda = 2$) are presently known and it is conjectured that there exist but finitely many symmetric designs for any fixed $\lambda \geqslant 2$. The authors know of fifteen biplanes.

The current state of affairs is as follows: The biplanes are unique for $k = 2$ through 5 [4]. There are precisely three for $k = 6$ [8]. As a result of an exhaustive computer search recently completed by the authors, there are precisely four biplanes for $k = 9$. We shall return to a discussion of these biplanes in section 3. There are two biplanes known for $k = 11$ and for $k = 13$. One of the biplanes with $k = 11$ was recently discovered by the authors with the aid of a computer and a description of the other can be found in [4]. The two for $k = 13$ are duals of each other and are described in [1]. There are no biplanes for $k = 7$, 8, 10, or 12 [4]. Essentially the same proof which demonstrates the non-existence of projective planes of orders congruent to 6 modulo 8 establishes the non-existence of biplanes with k congruent to 7 modulo 8 [2].

Our primary interest in biplanes stems from an algebraic coding theoretic connection between biplanes and projective planes which is briefly described as follows: If one precedes the incidence matrix of a biplane with k odd by a $v \times v$ identity matrix, then the rowspace of this $v \times 2v$ matrix is a self-dual code over GF(2) with minimum weight $k + 1$. One may sometimes extract a plane of order $k - 1$ from this rowspace by selecting the minimal-weight vectors with a 1 in some fixed coordinate. The planes of orders 2, 4, and 8

0097-3165/78/0242–0141$02.00/0

are obtainable in this manner. Unfortunately, the known biplanes with $k = 11$ and 13 fail to yield the putative planes of orders 10 and 12. Further information on this subject may be found in [2].

2. Definitions and Remarks

A *t-(v, k, λ) design* on a v-set S (a finite set of cardinality v whose elements are called points) is a collection, D, of k-subsets of S (called blocks) such that every t-subset of S is contained in precisely λ elements of D. The *automorphism group* of the design, $\operatorname{Aut}(S, D)$, is $\{\sigma \in \operatorname{Sym}(S) \mid \sigma \cdot B \in D$, for all $B \in D\}$. A *symmetric* design is one for which the number of blocks and points is the same. A *biplane*, $B(k)$, is a symmetric 2-$(v, k, 2)$ design. For a biplane, one easily shows that each point lies in k blocks, every pair of blocks have two points in common, and $k(k-1) = 2(v-1)$. The *dual* of a biplane is obtained by switching the roles of blocks and points.

We now present Hussain's manner of describing a biplane in terms of λ-chains [8]. Choose one of the blocks of the biplane and index its points from 1 to k. We shall call this block the *indexing block*. Since each pair of points occurs in precisely two blocks, each of the remaining blocks of the biplane is indexed by a unique 2-subset of the indexing block. Each point P not incident with the indexing block must occur in k blocks indexed by 2-subsets, and so we index each of these points by the collection of k 2-subsets. Each point incident with the indexing block must be an element of exactly two of the 2-subsets associated with P. If $\{a_1, a_2\}$, $\{a_2, a_3\}, \ldots, \{a_{n-1}, a_n\}$, $\{a_n, a_1\}$ is a collection of blocks incident with P, then we express this information by writing $(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$. The collection of cycles which represents such a point is called the *λ-chain* for the point. Hence we may represent each point not incident with the indexing block by a permutation on the k points of the indexing block, consisting of a product of disjoint cycles. It is clear that each cycle contains at least three letters. We shall call a λ-chain consisting of cycles containing $c_1, c_2, \ldots, c_m$ letters, respectively, a type $(c_1$–c_2– $\cdots$ –$c_m)$-chain. The *chain structure* of the biplane for a given indexing block is simply the number of chains of each type. Of course, unless the automorphism group of the biplane is transitive on its blocks, the chain structure may very well depend on the choice of the indexing block. Two biplanes are necessarily non-isomorphic if the collection of chain structures for one biplane is not identical to the collection for the other.

A λ-chain may also be viewed as a graph on the points of the indexing block, as Cameron does in [4], where the points become the vertices and the 2-subsets become the edges. Thus each λ-chain becomes a graph of valency 2 which is a disjoint union of polygons. A biplane is called *homogeneous* if the graphs for every choice of indexing block and non-incident point are all isomorphic. Such a biplane is of *characteristic* s if every graph is a disjoint union of s-gons, that is, every λ-chain consists of cycles, each containing s letters.

3. $B(9)$

An exhaustive search on the Lehigh University computer has established the existence of precisely four biplanes with $k = 9$, one of these being a new biplane. For $p \nmid (k - 2)$, the (mod p) rank of the incidence matrix of a biplane is a function of the parameters alone [6], and when $2 \neq p = k - 2$ is a prime, the (mod p) rank is $(v + 1)/2$ [9]. Thus the (mod p) rank necessarily fails to distinguish these biplanes.

One biplane with $k = 9$ can be constructed by letting the indexing block consist of the points of the projective line over GF(8) and indexing the remaining points of the biplane by the subgroups of order 3 of PSL(2, 8) [4]. Each λ-chain consists of three cycles, each being an orbit of the subgroup's action on the projective line. This biplane was originally discovered by Hussain [7] and we shall denote it by $B_H(9)$ and its non-isomorphic dual by $B_H'(9)$. The difference set biplane constructed via the biquadratic residues mod 37, independently discovered by Bose [3] and Fisher [5] will be denoted by $B_d(9)$, and the new biplane by $B_L(9)$.

We now list the collection of chain structures for each of these four biplanes. Of course, the only possible λ-chains are of type (9), (6-3), (5-4), and (3-3-3).

$B_H(9)$		$B_H'(9)$	
1 with 28	(3-3-3)-chains	9 with 28	(6-3)-chains
36 with 21	(9)-chains	28 with 27	(9)-chains
7	(6-3)-chains	1	(3-3-3)-chains

$B_d(9)$		$B_L(9)$	
37 with 19	(9)-chains	27 with 23	(9)-chains
9	(5-4)-chains	4	(6-3)-chains
		1	(5-4)-chains
		9 with 27	(9)-chains
		1	(6-3)-chains
		1 with 27	(9)-chains
		1	(3-3-3)-chains

Biplanes $B_H'(9)$ and $B_L(9)$ provide an example of two non-isomorphic biplanes having the same chain structure for particular choices of the indexing block.

Since $B_L(9)$ has a chain structure which occurs for the choice of only one indexing block, this block must be fixed by the automorphism group of the biplane. $\mathrm{Aut}(B_L(9))$ fixes a point not incident with the fixed block as well since this chain structure contains exactly one (3-3-3)-chain. We now present a description of $B_L(9)$ in terms of its automorphism group. $G = \mathrm{Aut}(B_L(9)) = \sum T\Psi = \{\sigma^i\tau^j\psi^k \mid \sigma^3 = \tau^3 = \psi^6 = 1,\ \tau\sigma = \sigma\tau,\ \psi\sigma = \sigma^{-1}\psi,\ \psi\tau = \tau^{-1}\psi^{-1},\ \psi\tau^{-1} = \sigma^{-1}\tau\psi^3\}$. Let $\mathrm{Syl}_3(G)$ be the Sylow-3 subgroup of G and let $\phi = \psi^2$, then $\mathrm{Syl}_3(G) = \sum T\Phi = \{\sigma^i\tau^j\phi^k \mid \sigma^3 = \tau^3 = \phi^3 = 1,\ \tau\sigma = \sigma\tau,\ \phi\sigma = \sigma\phi,\ \phi\tau = \sigma^{-1}\tau\phi\}$. Let $\{P_i\}$ and $\{B_i\}$, $1 \leqslant i \leqslant 3$, be base points and base blocks, respectively, for $B_L(9)$, then the collection of points and blocks can be represented by $\{P_1, P_2 \sum T, P_3 \sum T\Phi\}$ and $\{B_1, B_2 \sum T, B_3 \sum T\Phi\}$, respectively. $G_1 = G$, $G_2 = \Psi$ and $G_3 = \langle\psi^3\rangle$ where $G_i = G_{P_i} = G_{B_i}$, $1 \leqslant i \leqslant 3$. The incidence structure is defined by $B_1 = \{P_2 \sum T\}$,

$$B_2 = \{P_1, P_2\sigma, P_2\sigma^2, P_3\sigma^2\tau\Phi, P_3\sigma^2\tau^2\Phi\},$$

and

$$B_3 = \{P_2\sigma^2\tau, P_2\tau^2, P_3\sigma\tau^2, P_3\phi, P_3\tau\phi, P_3\sigma\tau^2\phi, P_3\sigma^2\tau^2\phi, P_3\tau\phi^2, P_3\sigma^2\tau\phi^2\}.$$

4. Concluding Remarks

The techniques employed in our exhaustive computer search for biplanes with $k = 9$ will appear in a forthcoming publication. Unfortunately, these methods are not powerful enough to exhaustively search for all biplanes with $k = 11$, or even for the seemingly most interesting class, namely, biplanes of characteristic 11. Since the method outlined in the Introduction for producing planes from biplanes works for biplanes of characteristics 3 and 5 and also works in the case of $B_H(9)$ by selecting the minimal-weight vectors with a 1 in the coordinate corresponding to the indexing block which produces all (3-3-3)-chains, we think that a biplane of characteristic 11 is the one most likely to have a connection with the putative plane of order 10.

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